**Thoughts on Green’s Functions**

**Interpretations**

Consider the construction of a GF in many-body physics. Its construction isolates the part of the energy levels due just to the excitation. So for instancewe can consider the green’s function as being kind of like cni(t) = <n|U(t)|i> where |i> = ci†|0> and |n> = cn†|0>. A difference is the extra U†(t) in the expression which factors out the time dependence of the |n> and the TC which gives it a pole structure). For consider,



Suppose that |0> and cn|0> are eigenstates of H, then we’d have,



and compare to cni under the same circumstances – where we inject an excitation at time t = 0 and extract it at time t = t.



So the green’s function isolates the energy shift (and lifetime) due to just the excitation part. And the θ(t) gives it a pole structure so that the excitation energy is the pole of g(ω).

The green’s function can be thought of as a probe, of creating an excitation ci† at time t = 0, and checking the probability that the state is in an excitation cn† by time t. But remember, cni (for whichever kind of perturbation) *does* calculate the actual transition probability from one state to another, overlap, etc. gni only heuristically does this. Proceeding a bit, we can calculate correlation functions, like,



where we have a state |n> at t = 0. Then we inject an excitation (or particle) into the system at t’, and measure the probability it will emerge in the excitation state at time t, which will give us the energy (just associated with the excitation) and lifetime of the state: c†|n>.

Note that finding Σ to first order is akin to finding ΔEc to first order, which is done at first order in RS, or BW perturbation theory. So each order of Σ corresponds to each order of RS, or BW PT. In order to go beyond to some sort of exact approximation, you’d have to sum an infinite # of Σ diagrams, which is done, or formulated, in self-consistent GF calculations. The decay/scattering of the excitation can be thought of as how equilibrium is maintained. According to the Drude formula excitations decay by 1/τk and we know now how to calculate this. So we expect the GF to look something like, GR(k,t) = -iZkexp(-i.θ(t). So consider for example the GF obtained when we make the c’s single particle creation annihilation operators. Note that if c is a particle annihilation operator, then it doesn’t excite the state, but rather adds a particle to it. Nonetheless this is a manner of excitation. And exactly as above, the form of the function isolates the change in energy of the system that is activated by the operator c† - that is by adding the particle to the non-interacting GS. Often times it seems that rather people attempt to purvey this information into information about particulate excitations above the true GS (according to Fermi-liquid theory, these should be fairly stable and be in 1-1 correspondence with the actual excitations of the system). It could be argued that if the system can support adding a particle near the Fermi-surface, then it would similarly support an excitation near the Fermi-surface. But in truth, that would technically be a different GF.

If we have strongly, strongly localized (interacting?) particles, then we expect position kets to be approximate eigenkets of the system. The degree to which this is the case can be measured by the correlation function:



This can be thought of as injecting a particle into the system at x, and measuring the likelihood basically that it is still there a time t later. If it were the case that position eigenkets were eigenstates of the system, then we ought to get



Taking the FT we’d have,



And we can observe from the pole structure that a position eigenket is a stationary state with energy V(x). Otherwise, if position eigenkets were close to being correct, we’d expect something like,



where Σ would shift the pole and the lifetime of the state, as Σ would (likely) be complex. Weakly interacting particles should occupy approximate single particle momentum states. The degree to which this is true is given by the following correlation function:



This can be thought of as injecting a plane wave state into the system and measuring its persistence a time t later. And we expect something like,



if momentum kets are eigenstates. Otherwise, we’ll get a self-energy deal which tells us how the energy shifts due to the interactions and how consequently the lifetime changes. And this would look like,



If particles interact with each other via HO potential, then we expect unperturbed eigenstates to be lattice vibrations (in other words density oscillations) – phonons. One situation where we might expect this to be the case is when/if e- became localized into a Wigner crystal. Then we’d expect excitations to be mainly oscillatory in nature. The degree to which this is the case can be measured by the correlation function,



This can be thought of as inducing a sound wave type motion into the system and measuring its persistence a time t later. We would expect something like,



(I suspect that this is why they choose the somewhat odd combination of a’s in the phonon Hamiltonian to create the propagator. I suspect this is because Ak­­ is ~ ρ(k). Indeed Ak­ ~ Uk which would be ρk if the phonon Hamiltonian that we work with in Ch. 5 had been 2nd quantized). Otherwise, we’ll get something like,



(you can remember the form because it is kind of like a foiled out square of difference). We could go further and ask whether these sorts of excitations are present, or make good approximations to eigenstates for higher energies. In other words, could you continue to add excitation to excitation to generate good approximations to higher eigenstates? Or a similar question is, do lattice oscillations continue to be an eigenstate even at high energies, or are lattice oscillations on the ground state (or other low energy states) the only energy window in which lattice oscillations form approximate eigenstates. We can answer this question by thermally averaging our expression above over all eigenstates of the system.



And so we can examine Σ as a function of T, and ω, k. Of course the trace will be centered about an energy ~ T per system, and an energy window of something. So we will basically be asking the same question as above, but with our answer parameterized by T, rather than n explicitly. A problem with this approach is that it requires a sum over all particle numbers as well – because it is hard to sum over only the section of FS with N particles, much easier to just sum over it all. But this probably wouldn’t affect things in a large N system, though a few body system probably would be poorly represented by such a technique.

It makes sense to ask about the structure of the system at temperature T. Of course the system has a fixed set of excitations defined only by H. But many of these excitations are such that as we raise the temperature, more and more are ‘activated’. For instance, water at low T, there would be mainly lattice oscillations. At high T would basically consist of particle like excitations. At low T, there are particle like excitations, but would require energies such that they are unlikely to be activated. So we would expect that at low T, the decay rate of the lattice oscillation GF would be low, whereas, the decay rate of the particle excitation GF would be high. So considering a system at temperature T, you’re considering the excitation structure of the system at the temperature T. I wonder what energy window this would correspond to? Note that one doesn’t have to define the trace to cover all particle numbers for the perturbative expansion to be valid – it works even if you just set it to cover only the FS with particle number N. The problem is that it is hard to calculate the GF0 when taking trace over just the N – space: unless N = 1. So the usual procedure is to take the trace over all N as well. If we want to restrict to certain N, then can set μ equal to such that ∂Ω/∂μ = N. This should suffice in most cases.

A potential problem with this GF approach is that you’ll miss certain excitations that can develop due to the interaction with other particles – for instance plasma oscillations against a jellium background which wouldn’t have been present otherwise. So a complete map of the excitations prevalent in this system isn’t just a renormalized account of the unperturbed system’s excitations. The SCHF method is also similar in spirit to this quasi-particle approach. So the GF method doesn’t look for the actual excitations of the system. It just asks how well the system supports a certain excitation. And when we ask how well it supports excitations of the unperturbed system, then we’re doing RS or BW perturbation theory. So just b/c the unperturbed system had only particle excitations, that doesn’t mean the perturbed system only has particle excitations – it might have plasma oscillations too, for instance.

So basically, the correlation function does the following. It takes an eigenstate, |n>, of the system, adds an excitation (so far discussed, an extra particle at position x, or an extra particle with momentum k, or an extra density oscillation with momentum k) at time t = 0, let’s the state then evolve for a time t more, and measures the overlap. If the |n + excitation> is an actual eigenstate of the system then it will continue to propagate unabated, and we get the GF above – if it is an approximate state then you get the energy shift and decay constant from Σ. The T = 0 GF uses |Ω> as |n>. The thermally averaged GF samples each eigenstate |n> with a weight e-βE. The non-equilibrium GF uses some other phase space. The qualitative nature of Σ can depend on T since T determines the occupation number of the single particle states nk and therefore the scattering phase space available. If there is little phase space available for scattering, then the test excitation may last much longer.

**Phase Conventions**

To determine what sort of U(t) to form, you ought to look at what you’re doing to the *state*. Suppose we have the Hamiltonian H = H0 + V and nothing depends on time. Then we make t0 = 0. For then, we’re describing a constant Hamiltonian and we should get the phase dependence on time of exp(-iεt), where ε is the exact eigenenergy. So for instance, we set up,



and this makes sense b/c consider the context under which we employ U(t). For suppose we say that we have a state |n> and that we then create a particle in it at time t’. And we want to measure its evolution up till time t. So we write,



and what we mean is that we start with }n> (at time t = 0 say). And then we create a particle at time t’. So we evolve the state to time t’, and then act with c†, and evolve back. And then we destroy at time t, so we evolve from t = 0 to time t, act with c, and then evolve back, to give us |n>. This is exactly how we would’ve described it had there been no ‘perturbation’. So this makes sense when written in terms of what we’re actually *doing* to the state we start with.

If we switch on V at time t0, then what we have depends on the situation. If we say that at time t = 0 we’re in the state |n>, then we’d use the same formula as above. For we would start at |n>, evolve forward from 0 to t’, create c† and evolve back. Then evolve forward to t, destroy c, and evolve back. So we’d still use:



However, we usually mean that we have the state |n> at the time t0 just before we turn on the perturbation. In that case we would use the U(t) below,



making similar arguments to previously. We start with |n> at t0 (so t0 is rather defined as when the system is in the state |n>) . Evolve forward to t′, create c† and evolve back. Then evolve forward to t to destroy c, and evolve back. So generally speaking, when dealing with a time dependent interaction we evolve from the time at which we know which state we’re in. We choose the appropriate U(t) to give us the required initial condition at t = t0 (and/or the required form for general t as in the time-independent case). Likewise, consider the evolution of ρ(t). We assume that at t0, just before the perturbation is switched on, that ρ(t) = ρeq.(t) = eiH0tρeq.e-iH0t. So it all depends on what your initial conditions are and when they are. And you can look to either the states your operating on, or the operator to give clues as to what the IC are. And you must think about what it is that you’re doing, or how you’re measuring it. It is senseless to ask what ψ(t) is without specifying an initial condition, and it is senseless also to ask what a†(t) is without specifying an initial condition. You *have* to.